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Supersymmetry and the Ladder Operator Technique in Quantum Mechanics: The Radial Schrödinger Equation

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Using the ladder operator technique, a construction of the supersymmetric Hamiltonian is proposed. We show that the accidental degeneracies associated with the Coulomb and isotropic oscillator problems may be attributed to the existence of a supersymmetry of the Hamiltonians.

1. INTRODUCTION

The connection between the Schrödinger factorization method (Schrödinger, 1940, 1941) and supersymmetric quantum mechanics (Witten, 1981) has been developed (see, for instance, Cooper and Freedman, 1983) and explored (see, for instance, Salmonson and van Holten 1982) by many authors. As is well known, the idea of supersymmetry (SUSY) is to relate integral spin objects to half-integral ones, and quite interestingly, all its essential features are present (Ravndal, 1984) in field theories of $(1+0)$ dimensions.

The algebraic properties of the factorization method and their link to SUSY rest on the most important fact that if the ground-state wave function is known completely, then the factorization of the Hamiltonian follows as a natural consequence (Kwong and Rosner, 1986). Indeed, this is how SUSY in quantum mechanics has been formulated in terms of the factorization method, and its relationship to elementary quantum mechanical

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systems has been investigated (Gendenshtein and Krive, 1985; Sukumar, 1985, 1986). It is also not difficult to see from this work that the factorizability criterion and supersymmetric properties are common to all one-dimensional potentials.

The question we raise in this paper is how to carry out judiciously the procedure of factorization for three-dimensional potentials and construct partner Hamiltonians, which are related by SUSY. That SUSY may work differently for radial problems has been noted by a number of authors (Haymaker and Rau, 1986; Lahiri *et al.*, 1987). It should be clear from this work that since the radial problem is not truly one dimensional, the construction as well as the interpretation of SUSY are problematical. The standard one-dimensional approach of writing down the superpotential and reading off energy degeneracies from the partner Hamiltonians may lead to fallacious results. From this point of view, the method of Kostelecky and Nieto (1984) has been questioned by Haymaker and Rau (1986).

It may be noted that Kostelecky and Nieto have proposed a supersymmetric construction of the hydrogen atom by subjecting the radial part of the Schrödinger equation to a one-dimensional treatment. As a consequence, they have found that the supersymmetric partner Hamiltonians describe the hydrogenic *ns–sp* degeneracy. They have thus interpreted their results as giving a supersymmetric connection between various atoms. On the other hand, Haymaker and Rau have advocated a transformation of the half-line problem to the full-line one, thereby passing from the radial to the Morse problem. As a result, their scheme[4] yields a connection between isoelectronic ions, in contrast to those between states of different atoms.

In this work, we propose a method for the construction of the supersymmetric Hamiltonian to account for the above-mentioned "accidental" degeneracies. Employing the ladder operator techniques, we have developed a scenario which is straightforward and avoids complicated manipulations of the differential equations. We have considered the radial Coulomb and the isotropic oscillator problems and have found that the accidental degeneracies associated with them may be connected with the existence of SUSY of the governing Hamiltonian. In particular, we have shown that SUSY can account for the *ns–np* degeneracy of the Coulomb problem, which is consistent with similar claims by Kostelecky and Nieto (1984) and Kostelecky *et al.* (1985). We should mention that throughout this work, we have avoided any reference to the superpotential, but have formulated a self-consistent scenario involving lowering and raising operators. Indeed, this is a convenient framework in which the standard difficulties with the three-dimensional construction of SUSY are avoided.

[4]The approaach of Haymaker Rau is beset with some difficulties [in this regard see Lahiri *et al.* (1987)].

2. SUSY HAMILTONIAN

Since the degeneracy due to SUSY arises by simultaneously destroying one bosonic (b) quantum and creating one fermionic (a) quantum (or vice versa), the corresponding generators behave like ba^+ or b^+a.

One may define SUSY generators as

$$Q^+ = \sqrt{(2\omega)}\, ba^+, \qquad Q^- = \sqrt{(2\omega)}\, b^+a \tag{1}$$

both of which commute with the SUSY Hamiltonian

$$H_{ss} = (\omega/2)(a^+a + b + b)$$

It may be verified that Q^+ and Q^- as defined in (1) satisfy

$$[Q^+, H_{ss}] = [Q^-, H_{ss}] = 0$$

Also, one has

$$\{Q^+, Q^+\} = \{Q^-, Q^-\} = 0$$

or

$$(Q^+)^2 = (Q^-)^2 = 0$$

While the bosonic creation and annihilation operators can be represented in terms of q and p, the corresponding ones for the fermionic system can be represented in terms of the Pauli matrices:

$$a = \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad a^+ = \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

where

$$\{\sigma_-, \sigma_+\} = \mathbb{1}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

In this representation the SUSY Hamiltonians can be written as

$$\begin{aligned} H_{ss} &= (\omega/2) b^+b\mathbb{1} + (\omega/2)\sigma_3 \\ &= \frac{\omega}{2}\begin{pmatrix} bb^+ & 0 \\ 0 & b^+b \end{pmatrix} \\ &= \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix} \end{aligned} \tag{2}$$

3. METHOD OF FACTORIZATION

For a spherically symmetric potential the Schrödinger equation separates into a radial and an angular part. The latter may be subjected to a

SUSY construction. The reduced radial equation with $V(r)$ having a rotation symmetry reads ($\hbar = 2m = 1$)

$$H_l|nl\rangle = E_l^n|nl\rangle$$

where

$$H_l = p^2 + l(l+1)/r^2 + V(r)$$

is the standard radial Hamiltonian. We label the eigenket $|nl\rangle$ and the energy eigenvalue E_l^p by the radial quantum number n and the angular momentum quantum number l.

To find a suitable form of the ladder operator A which maps $|nl\rangle$ onto $|n'l'\rangle$ requires (Newmarch and Golding, 1978) an effective factorization of the Hamiltonian. We set

$$\begin{aligned} A_l^+ A_l &= H_l + F_l \\ A_l A_l^+ &= H_{l'} + G_{l'} \end{aligned} \tag{3}$$

where F_l and $G_{l'}$ aare just scalar objects and the operators A_l and A_l^+ are to be defined presently. Before we do so, let us note that the quantity $A_l A_l^+ A_l|nl\rangle$ may be evaluated, using (3), as

$$\begin{aligned} A_l A_l^+ A_l|nl\rangle &= (E_l^n + F_l)A_l|nl\rangle \\ &= (H_{l'} + G_{l'})A_l|nl\rangle \end{aligned}$$

Rearranging, one can write

$$H_{l'}(A_l|nl\rangle) = \{E_l^n + (F_l - G_{l'})\}(A_l|nl\rangle)$$

which expresses that $A_l|nl\rangle$ is an eigenket of $H_{l'}$ with eigenvalue

$$E_{l'}^{n'} = E_l^n + (F_l - G_{l'})$$

If $F_l = G_{l'}$, we have a degeneracy between states with quantum numbers n, l and n', l'. To fix n' and l', we choose the shifts $n' = n - \mu$ and $l' = l + \lambda$, with μ and λ being any integer. That is,

$$E_{l+\lambda}^{n-\mu} = E_l^n \tag{4}$$

We shall soon see that n' gets fixed as $n' = n - 1$.

From equation (4), we then have

$$\begin{aligned} A_l^+ A_l &= H_l + F_l \\ A_l A_l^+ &= H_{l+\lambda} + F_l \end{aligned} \tag{5}$$

We now define the operators A_l and A_l^+ as follows:

$$\begin{aligned} A_l|nl\rangle &= \alpha_l^n|n-\mu, l+\lambda\rangle \\ A_l^+|n-\mu, l+\lambda\rangle &= \beta_l^n|nl\rangle \end{aligned} \tag{6}$$

where α_l^n and β_l^n are some constants. Since

$$\alpha_l^n = \langle n-\mu, l+\lambda | A_l | nl \rangle$$
$$\beta_l^n = \langle nl | A_l^+ | n-\mu, l+\lambda \rangle$$

we conclude that the definitions (6) imply that α_l^n and ρ_l^n are complex conjugates of one another: $\alpha_l^n = \beta_l^{n*}$. In other words, the operators $A_l^+ A_l$ and $A_l A_l^+$ have the same eigenvalue $|\alpha_l^n|^2$.

Now, this eigenvalue $|\alpha_l^n|^2$ may be obtained from (3),

$$A_l^+ A_l |nl\rangle = (H_l + F_l)|nl\rangle$$

or

$$|\alpha_l^n|^2 = E_l^n + F_l$$

Noting that the operation of A_l on $|nl\rangle$ decreases the radial quantum number by μ, we can construct a sequence of eigenkets by repetitive applications of A_l on $|nl\rangle$:

$$A_l|nl\rangle = \alpha_l^n |n-\mu, l+\lambda\rangle$$
$$(A_l)^2|nl\rangle = \alpha_l^n \alpha_l^{n-\mu} |n-2\mu, l+2\lambda\rangle$$
$$(A_l)^m|nl) = \alpha_l^n \alpha_l^{n-\mu} \cdots \alpha_l^{n-(m-1)\mu} |n-m\mu, l+m\lambda\rangle$$

where m stands for the number of times A_l has been applied. Since the above sequence must terminate for a spectrum having a lower bound, we may define $A_l|0, l\rangle = 0$. It is obvious that in such a case $m = n/\mu$. Also, since the power of A_l must be an integer (m being related to the number of applications of A_l), consistency with the allowed values of n requires that μ be set equal to unity. It may be noted that the possible values n can take are $0, 1, 2, \ldots$. Relation (4) is thus modified to

$$E_{l+\lambda}^{n-1} = E_l^n \tag{4a}$$

Further, since $A_l|0, l\rangle = 0$, we have $\alpha_l^0 = 0$. This fixes F_l as $F_l = -E_l^0$, so that

$$|\alpha_l^n|^2 = E_l^n - E_l^0$$

We can now find a connection with the supersymmetric Hamiltonian (2). Since, from (6), one may identify A_l and A_l^+ as lowering and raising operators, respectively, we can construct the generators Q^+ and Q^- in terms of A_l and A_l^+:

$$Q^+ = \sigma_+ A_l = \begin{pmatrix} 0 & A_l \\ 0 & 0 \end{pmatrix}$$

$$Q^- = \sigma_- A_l^+ = \begin{pmatrix} 0 & 0 \\ A_l^+ & 0 \end{pmatrix}$$

The Hamiltonian is then given by

$$H_{ss} = \{Q^+, Q^-\} = \begin{pmatrix} A_l A_l^+ & 0 \\ 0 & A_l^+ A_l \end{pmatrix} = \begin{pmatrix} H^- & 0 \\ 0 & H^+ \end{pmatrix}$$

Using (5), the SUSY partner Hamiltonians for a spherically symmetric problem may be obtained as

$$\begin{aligned} H^+ &= A_l^+ A_l = H_l - E_l^0 \\ H^- &= A_l A_l^+ = H_{l+\lambda} - E_l^0 \end{aligned} \tag{7}$$

The above forms for H^+ and H^- do not seem to have been recognized in the context of SUSY before. With (7) at hand, we are in a position to turn to specific problems.

4. APPLICATIONS

4.1. Coulomb Potential: $V(r) = -1/r$

The energy eigenvalues are given by

$$E_l^n = -(n+l+1)^{-2}/4 = -1/(4N^2)$$

where N is the principal quantum number. λ should be obtained from the relation (4a). We have

$$(n-1+l+\lambda+1)^{-2}/4 = (n+l+1)^{-2}/4$$

or $\lambda = 1$. Hence

$$H^+ = H_l - E_l^0 = p^2 + l(l+1)/r^2 - 1/r + (l+1)^{-2}/4$$
$$H^- = H_{l+\lambda} - E_l^0 = p^2 + (l+1)(l+2)/r^2 - 1/r + (l+1)^{-2}/4$$

Thus it becomes obvious that the energy degeneracy due to SUSY is between states of n, l and $n-1$, $l+1$. This is the so called *ns*-*np* degeneracy.

4.2. Isotropic Oscillator Potential: $V(r) = r^2/4$

The energy eigenvalues are given by (Baumgartner *et al.*, 1985)

$$E_l^n = 2n + l + \tfrac{3}{2}$$

Relation (4a) gives $\lambda = 2$. Hence, from (7),

$$H^+ = H_l - E_l^0 = p^2 + l(l+1)/r^2 + r^2/4 - (l+\tfrac{3}{2})$$
$$H^- = H_{l+\lambda} - E_l^0 = p^2 + (l+2)(l+3)/r^2 + r^2/4 - (l+\tfrac{3}{2})$$

We thus find that the energy degeneracy due to SUSY is between states of quantum numbers (n, l) and $(n-1, l+2)$.

5. SUMMARY

We have prescribed in a simple way the construction of a supersymmetric Hamiltonian with the help of bosonic and fermionic oscillators. We have also discussed the ladder operator technique which may be used for such a construction. The method developed has been applied to the case of the Coulomb and isotropic oscillator problems. We have shown that their accidental degeneracies may be attributed to the SUSY of the Hamiltonian.

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